Generating Functions – An Introduction

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Motivation

- Bridge between discrete mathematics and analysis. How can the continuous way be helpful to understand discrete problems.
- Powerful tools for enumeration problems – counting the number of objects of size n satisfying a condition. e.g. *How many binary sequences are there of length n?*
- In some cases, they can provide shorter proofs for questions.
- At a first glace, especially in some problems that we will see, it might seem that this approach is tedious, however note that the purpose of this lecture is to only introduce the concept and this topic goes much deeper in the form of *Analytic Combinatorics*.
- These tools can be applied to a large number of problems in discrete mathematics including permutations, trees, graphs, etc.

Preliminaries

- A *combinatorial class*, $\mathcal{A}$ is a countable set, on which a *size* function is defined satisfying the following conditions:
  1. size of any element is a non negative integer.
  2. the number of elements of any given size is finite.
- A counting sequence for $\mathcal{A}$ is a sequence of integers $(A_n)_{n\geq 0}$ where each A_n is the number of objects in $\mathcal{A}$ of size n.
- **Example**. For $\mathcal{A} = \{\varepsilon, 0, 1, 00, 01, 10, 11, 100, 101, 110, 111, \ldots\}$. If size is length of the binary string then, the counting sequence is $(2^n)_{n\geq 0}$.
- Example. *Permutations*: A permutation of size n is a bijective mapping on $\{1, 2, \ldots, n\}$. The combinatorial class here is the set of all permutations over $n \geq 0$, i.e.

$$\mathcal{A} = \{\ldots, 1, 12, 21, 123, 132, 213, 231, 312, 321, 1234, 1243, \ldots\}$$

The counting sequence here is $(n!)_{n\geq 0}$

Ordinary Generating Function

Definition

The ordinary generating function of a sequence $(a_n)_{n\geq 0}$ is given by the *formal* power series $A(z) = \sum_{n\geq 0} A_n z^n$.

- Note that formal power series is not the same as a power series, we ignore questions of convergence.
- z does not hold any numerical value and arithmetic operations are carried out by pretending that they are polynomials.
- Two fundamental operations like addition and multiplication hold here, through which the result $\frac{1}{1-z} = 1 + z + z^2 + \ldots$ holds.
- For a combinatorial class, $\mathcal{A}$, the ogf $\sum_{n\geq 0} A_n z^n$ is the generating function of the counting sequence.
- Example: if $\mathcal{A}$ is power set of $\{1, 2, \ldots k\}$. Then we know that the number of subsets of size $n \leq k$ is $\binom{k}{n}$. The corresponding generating function will be $A(z) = \sum_{k=0}^{n} \binom{k}{n} z^k = (1+z)^k$.

Basic Properties

- *Disjoint Union*: If $\mathcal{A}$ are $\mathcal{B}$ are two disjoint combinatorial classes then if $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, then the generating function of the counting sequence of $\mathcal{C}$ is given by $C(x) = A(x) + B(x) = \sum_{n \geq 0}(A_n + B_n)x^n$.
- *Cartesian Product*: If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, i.e. set of all (a, b) with $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then $C(x) = A(x) \cdot B(x)$, or $C_n = \sum_{i=0}^{n} a_i b_{n-i}$.

Example. How many ways can one obtain the sum n from two 6 faced die?

- Without the knowledge of ogfs, we can have $n = 7$, using $16, 25, 34, 61, 52, 43$, which is essentially the same as $\sum_{i \geq 0} a_i a_{7-i}$, where a_i is the number of ways of obtaining the number i.
- Now, from the method of ogfs, we write $\mathcal{C}$ as $\mathcal{A} \times \mathcal{A}$, where $\mathcal{A}$ is the set $\{1, 2, 3, 4, 5, 6\}$, and the corresponding generating function is $x + x^2 + x^3 + x^4 + x^5 + x^6$, where A_n is the number of times n is in $\mathcal{A}$. Here, the generating function is $(\sum_{i=1}^{6} x^i)^2$, and simply the coefficient of x^n will give us the answer.

Coin change problem

Problem

Given a 5 pennies (1 cent),2 nickels (5 cents) and 2 dimes (10 cents), how many ways can we make an exact change of n cents?

- We have seen this in the form of dynamic programming, where $A_{n,coin_i} = A_{n-1} + A_{n-coin_i}$.
- The generating function for this problem becomes

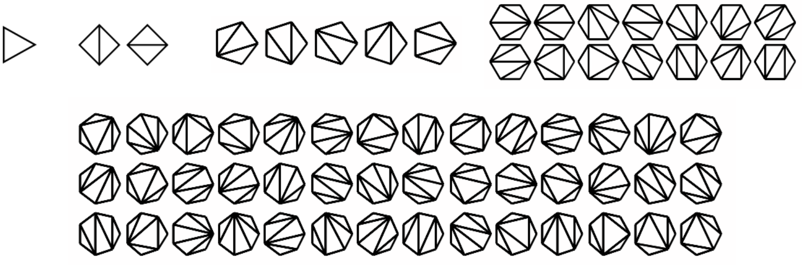
$$G(x) = (1 + x + x^2 + \ldots x^5) \cdot (1 + x^5 + x^{10}) \cdot (1 + x^{10} + x^{20})$$

- Of course, this is also not easy to calculate without Wolfram.
- What if we are allowed an unlimited number of pennies, nickels and dimes? $G(x) = \frac{1}{(1-x)} \cdot \frac{1}{(1-x^5)} \cdot \frac{1}{(1-x^{10})}$.

Triangulation

Problem

Given a k sided polygon, how many ways are there of putting chords to divide it into triangles?



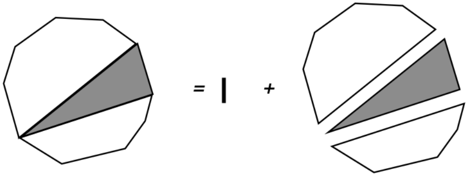
We take $T_1 = T_2 = 1$. We can see that $T_3 = 1$, $T_4 = 2, T_5 = 5, \ldots$

Triangulation

- The recurrence relation is given as

$$T_{k-2} = \sum_{j=2}^{n-2} T_{j-2} T_{k-j-1} \quad \text{OR} \quad T_k = \sum_{j=0}^{k-1} T_j T_{k-j-1}$$

- This is well known ...
- We can use gfs to get the exact formulation.
- Expanding from the sequence, we can obtain that $T(x) = 1 + x \cdot T^2(x)$.
- Another way to see this is $\mathcal{T} = \{\varepsilon\} + (\mathcal{T} \times \Delta \times \mathcal{T})$.



Triangulation

- From a simple quadratic formula, we obtain $T(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. To obtain $T(0) = 1$, we only take the $-$ term.
- We still need coefficients of x^n.
- A tedious process follows where we expand $(1-4x)^{1/2}$ using some weird formal power series properties like

$$(1-4x)^{1/2} = 1 - \binom{\frac{1}{2}}{1} y + \binom{\frac{1}{2}}{2} y^2 + \dots$$

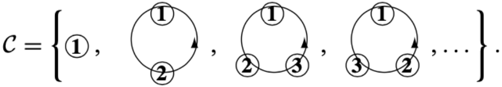
- After some normal algebra, we obtain $T_n = \frac{1}{n+1}\binom{2n}{n} x^n$ which is the Catalan Number!

Exponential Generating Function

Definition

An exponential generating function (egf) of a sequence (A_n) is the formal power series $A(z) = \sum_{n\geq 0} A_n \frac{z^n}{n!}$.

- These functions are useful for the case where elements are distinguishable from one another, like in graphs, trees, permutations.
- **Example**: The class of all permutations that we saw is a prototypical labelled class. The egf of this combinatorial class is $P(z) = \sum_{n\geq 0} n! \frac{z^n}{n!} = \frac{1}{1-z}$.
- **Example**: The class of cyclic permutations.



$$C(z) = \sum_{n\geq 0}(n-1)!\frac{z^n}{n!} = \sum_{n\geq 0} \frac{z^n}{n} = \log\left(\frac{1}{1-z}\right).$$

- Observation: $P(z) = e^{C(z)}$.

Permutations

Theorem

Any permutation can be described in terms of a disjoint collection of cycles.

- Suppose on the set $\{1, 2, 3, 4, 5, 6\}$, $\pi(1) = 2, \pi(2) = 5, \pi(3) = 6, \pi(4) = 4, \pi(5) = 1, \pi(6) = 3$.
- The permutation can be described by $1 \to 2 \to 5 \to 1$, $3 \to 6 \to 3$ and $4 \to 4$.

Permutations

Theorem

Any permutation can be described in terms of a disjoint collection of cycles.

- To enumerate all permutations, we consider a all partitions of the set, S, and for each partition we consider a cycle.
- First, what is the number of permutations of a size of size n with exactly k cycles?
- Ans. $\frac{C(x)^k}{k!}$.
- For $k = 2$, the number of ways of partitioning S into two cycles is

$$q_n = \sum_{T \subset S} C_{|T|} \cdot C_{n-|T|} = \sum_{t=0}^{n} \binom{n}{t} C_t C_{n-t}$$

, where C_n is the counting sequence of a cycle.

Permutations

- Writing the egf, will give us

$$\begin{aligned} Q(x) = \sum_{n\geq 0} \frac{q_n}{n!} \cdot x^n &= \sum_{n\geq 0} \frac{1}{n!} \cdot \left(\sum_{t=0}^{n} \frac{n!}{t!(n-t)!} c_t c_{n-t} \right) x^n \\ &= \sum_{t=0}^{\infty} \sum_{n=t}^{\infty} \frac{c_t x^t}{t!} \cdot \frac{c_{n-t} x^{n-t}}{t!} \\ &= C(x) \cdot C(x) \end{aligned}$$

- We have counted each permutation twice, hence our answer is $\frac{C(x) \cdot C(x)}{2!}$.

Permutations

- Similarly we can show that for k cycles, the egf is $C(x)^k$ and the coefficient of n is

$$q_n = \sum_{n_1, n_2, \ldots n_k, \sum_i n_i = n} \frac{c_{n_1}}{n_1!} \cdots \frac{c_{n_k}}{n_k!} = \frac{1}{n!} \sum_{n_1, n_2, \ldots n_k, \sum_i n_i = n} \binom{n}{n_1 \quad n_2 \ldots n_k} c_{n_1} \ldots c_{n_k}$$

- The proof of $P(x) = e^{C(x)}$ can be seen by taking partitions of all sizes $k = 1, 2, \ldots, n$ and applying a Taylor Series (which is valid for formal power series).

$$P(x) = \sum_{k=1}^{n} \frac{1}{k!} \cdot (C(x))^k = e^{C(x)}.$$

An incomplete list of problems solvable with gfs

- Fibonacci recurrence
- Quicksort recurrence
- Birthday Paradox
- Coupon collector
- Diagonal sum of pascal's triangle
- Deragements.

References

- Analytic Combinatorics- Flajolet and Sedgewick.
- Introduction to Analysis of Algorithms - Flajolet and Sedgewick.
- Notes on Generating Functions by Michel Goemans.
- generatingfunctionology by herbert S. Wilf.